A Novel Class of Localized Excitations for the (2+1)-Dimensional Higher-Order Broer-Kaup System

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By applying a special Bäcklund transformation, a general variable separation solution for the (2+1)-dimensional higher-order Broer-Kaup system is derived. In addition to some types of the usual localized excitations, such as dromions, lumps, ring solitons, oscillated dromions and breathers, soliton structures can be easily constructed by selecting arbitrary functions appropriately. A new class of localized structures, like fractal-dromions, fractal-lumps, peakons, compactons and folded excitations of this system is found by selecting appropriate functions. Some interesting novel features of these structures are revealed. — PACS numbers: 05.45.-a, 02.30.Jr, 02.30.Ik.

Key words: Extended Homogeneous Balance Method; Variable Separation Solution; (2+1)-Dimensional HBK System; Localized Excitations.

1. Introduction

Nonlinear partial differential equations are widely used to describe complex phenomena in biology, chemistry and mathematics, and especially in physics [1]. Recently several significant (2+1)-dimensional models [2, 8 – 14] have been investigated, and some special types of localized solutions for these models have been obtained by means of different approaches (VSA), for example the bilinear method, standard and extend truncated Painleve analysis, variable separation approach, standard and extended homogeneous balance method, and so on [3 – 5]. From these studies of (2+1)-dimensional models one can see that there exist more abundant localized structures than in lower dimensions. This implies that there may exist similar or new localized structures that are unrevealed in other (2+1)-dimensional integrable models. In this paper, we further consider the (2+1)-dimensional higher-order Broer-Kaup (HBK) system

$$H_t = -4(H_{xx} + H^3 - 3HH_x + 3H\partial_y^{-1}V_x + 3\partial_y^{-1}(VH)_x)_x, \tag{1a}$$

$$V_t = -4(V_{xx} + 3V_xH + 3H^2V + 3V\partial_y^{-1}V_x)_x, \tag{1b}$$

which is obtained from the Kadomtsev-Petviashvili (KP) equation by symmetry constraints [6]. If we take $y = x$, the system can be degenerated to the (1+1)-dimensional HBK equation

$$H_t = -4(H_{xx} + H^3 + 6HV - 3HH_x)_x, \tag{2a}$$

$$V_t = -4(V_{xx} + 3V_xH + 3H^2V + 3V^2)_x. \tag{2b}$$

Starting from a special Bäcklund transformation obtained by using the extended homogeneous balance method (EHBM) [4] and the VSA [5], we convert the HBK system into a simple variable separation equation, and then obtain a quite general solution. For some types of the usual localized excitations of (1), such as dromions, lumps, ring solitons and oscillated dromions, breathers solutions can be easily constructed by selecting appropriate arbitrary functions. In addition to the usual localized structures, some new localized excitations, like fractal-dromions, fractal-lumps, peakons, compactons, folded solitary waves and foldon solutions of (1), are found by selecting some types of lower-dimensional appropriate functions. Moreover, it has been found that fractal-dromions and lumps, peakons, compactons and foldons may have many interesting properties and possible physical applications [7 – 14]. But for these lower-dimensional fractal-dromions and lumps, peakons, compactons and foldons we know little in higher dimensions.

2. Variable Separation Solution Based on the Extended Homogeneous Balance Method for the (2+1)-Dimensional HBK System

In order to get the special solution of model (1), we rewrite (1) in the following potential form:

$$\begin{aligned} &H_t + 4(H_{xx} + H^3 - 3HH_x + 3HW + 3P)_x = 0,\\ &V_t + 4(V_{xx} + 3V_xH + 3H^2V + 3VW)_x = 0,\\ &W_y - V_x = 0, \quad P_y - (VH)_x = 0. \end{aligned} \tag{3}$$

by using $W = \partial_y^{-1}V_x$, $P = \partial_y^{-1}(VH)_x$.

According to the EHBM, let

$$\begin{aligned} H &= f_x(\phi(x,y,t)) + H_0(x,y,t),\\ V &= f_{xy}(\phi(x,y,t)) + V_0(x,y,t),\\ W &= f_{xx}(\phi(x,y,t)) + W_0(x,y,t),\\ P &= f_x(\phi(x,y,t))f_{xx}(\phi(x,y,t)) + P_0(x,y,t), \end{aligned} \tag{4}$$

where H_0, V_0, W_0, and P_0 are the arbitrary solutions of the (2+1)-dimensional HBK system. This means that (4) is a Bäcklund transformation of the (2+1)-dimensional HBK system. For convenience, we fix the original seed solution as

$$H_0 = V_0 = 0, \quad W_0 = W_0(x,t), \quad P_0 = C. \tag{5}$$

Introducing (4) and (5) into (3), we obtain

$$\begin{aligned} &H_t + 4(H_{xx} + H^3 - 3HH_x + 3HW + 3P)_x\\ &\quad = (12f''^2 + 12f'f^{(3)} + 4f^{(4)} + 12f'^2f'')\phi_x^4 \end{aligned} \tag{6a}$$

and

$$\begin{aligned} &V_t + 4(V_{xx} + 3V_xH + 3H^2V + 3VW)_x\\ &\quad = (4f^{(5)} + 12f'^2g^{(3)} + 24f'f''^2\\ &\qquad + 36f''f^{(3)} + 12f'f^{(4)})\phi_y\phi_x^4 \end{aligned} \tag{6b}$$

+ lower power terms of the derivatives of $\phi(x,y,t)$ with respect to x, y, and t.

Setting the coefficients of ϕ_x^4 in (6a) and $\phi_y\phi_x^4$ in (6b) to zero yields the ordinary differential system

$$12f''^2 + 12f'f^{(3)} + 4f^{(4)} + 12f'^2f'' = 0, \tag{7a}$$

$$\begin{aligned} &4f^{(5)} + 12f'^2g^{(3)} + 24f'f''^2\\ &\quad + 36f''f^{(3)} + 12f'f^{(4)} = 0. \end{aligned} \tag{7b}$$

The following special solutions exist for (7):

$$f(\phi) = \ln(\phi). \tag{8}$$

Using the above results, (6) can be simplified as

$$\begin{aligned} &H_t + 4(H_{xx} + H^3 - 3HH_x + 3HW + 3P)_x\\ &= (4\phi_{xxx} + 12\phi_xW_0 + \phi_t)_x f'\\ &\quad + [(4\phi_{xxx} + 12\phi_xW_0 + \phi_t)\phi_x]f'' = 0, \end{aligned} \tag{9a}$$

$$\begin{aligned} &V_t + 4(V_{xx} + 3V_xH + 3H^2V + 3VW)_x\\ &= (4\phi_{xxx} + 12\phi_xW_0 + \phi_t)_{xy} f'\\ &\quad + \Big\{(\phi_x(4\phi_{xxx} + 12\phi_xW_0 + \phi_t))_y\\ &\qquad + \phi_y(4\phi_{xxx} + 12\phi_xW_0 + \phi_t)_x\Big\}f''\\ &\quad + \{\phi_x\phi_y(4\phi_{xxx} + 12\phi_xW_0 + \phi_t)\}f^{(3)} = 0. \end{aligned} \tag{9b}$$

Setting the coefficients of $f^{(3)}$, f'', f' in (9) to zero and simplifying yields a set of partial differential equations for $\phi(x,y,t)$:

$$(4\phi_{xxx} + 12\phi_xW_0 + \phi_t)_x = 0, \tag{10a}$$

$$(4\phi_{xxx} + 12\phi_xW_0 + \phi_t)\phi_x = 0, \tag{10b}$$

$$(4\phi_{xxx} + 12\phi_xW_0 + \phi_t)_{xy} = 0, \tag{10c}$$

$$\begin{aligned} &(\phi_x(4\phi_{xxx} + 12\phi_xW_0 + \phi_t))_y\\ &\quad + \phi_y(4\phi_{xxx} + 12\phi_xW_0 + \phi_t)_x = 0, \end{aligned} \tag{10d}$$

$$\phi_x\phi_y(4\phi_{xxx} + 12\phi_xW_0 + \phi_t) = 0. \tag{10e}$$

Analyzing the above equations, we find that (10a)–(10e) are satisfied automatically if

$$4\phi_{xxx} + 12\phi_xW_0 + \phi_t = 0. \tag{11}$$

For the linear equation (11) of the original system we can construct many types of special solutions. Because W_0 is an arbitrary function with respect to the variables $\{x,t\}$, ϕ and W_0 in (11) have the separated variable solutions

$$\phi = \alpha_1 + \alpha_2\beta, \quad W_0(x,t) = -\frac{4\beta_{xxx} + \beta_t}{12\beta_x}, \tag{12}$$

where $\beta \equiv \beta(x,t)$ is an arbitrary function of the variables $\{x,t\}$, and $\alpha_1 \equiv \alpha_1(y)$ and $\alpha_2 \equiv \alpha_2(y)$ are functions of y. Introducing (5), (8), and (12) into (4), we have

$$H = \frac{\alpha_2\beta_x}{\alpha_1 + \alpha_2\beta}, \tag{13a}$$

$$V=\frac{\alpha_{2y}\beta_x}{\alpha_1+\alpha_2\beta}-\frac{\alpha_2\beta_x(\alpha_{1y}+\alpha_{2y}\beta)}{(\alpha_1+\alpha_2\beta)^2}, \tag{13b}$$

$$W=\frac{\alpha_2\beta_{xx}}{\alpha_1+\alpha_2\beta}-\frac{\alpha_2^2\beta_x^2}{(\alpha_1+\alpha_2\beta)^2}-\frac{4\beta_{xxx}+\beta_t}{12\beta_x}, \tag{13c}$$

$$P=\frac{\alpha_2^2\beta_x\beta_{xx}}{(\alpha_1+\alpha_2\beta)^2}-\frac{\alpha_2^3\beta_x^3}{(\alpha_1+\alpha_2\beta)^3}+C. \tag{13d}$$

Because of the arbitrariness of the functions α_1, α_2, and β in (13), the solutions of the (2+1)-dimensional HBK possess quite rich structures. In the next section we focus on some new and interesting special examples, such as fractal-dromions, fractal-lumps, compactons, peakons and folded localized excitations and their interaction behavior.

3. Some New Localized Excitations of the (2+1)-Dimensional HBK System

3.1. Fractal Dromions and Lumps Localized Excitations

For (2+1)-dimensions we know that among the most important localized excitations are the so-called dromion solutions which are exponentially localized in all directions. Recently it was found that many lower-dimensional piecewise smooth functions with fractal structures can be used to construct exact localized solutions of higher-dimensional soliton systems which also possess fractal structures [15]. This situation also occurs in the (2+1)-dimensional HBK system. If we select both α_1, α_2 and β as some types of fractal functions appropriately, we may obtain some special types of fractal dromion solutions. We call a dromion solution a fractal dromion if the solution is exponentially localized in a large scale and possesses a self-similar structure near the dromion centre. For instance, if we take

$$\alpha_1(y)=1, \tag{14a}$$

$$\alpha_2(y)=1+\exp\{-y[y+\sin(\ln(y)^2)+\cos(\ln(y)^2)]\}, \tag{14b}$$

$$\beta(x,t)=1+\exp\{-x[x+\sin(\ln(x-\gamma t)^2)+\cos(\ln(x-\gamma t)^2)]\}, \tag{14c}$$

the field quantity V becomes a special fractal dromion solution.

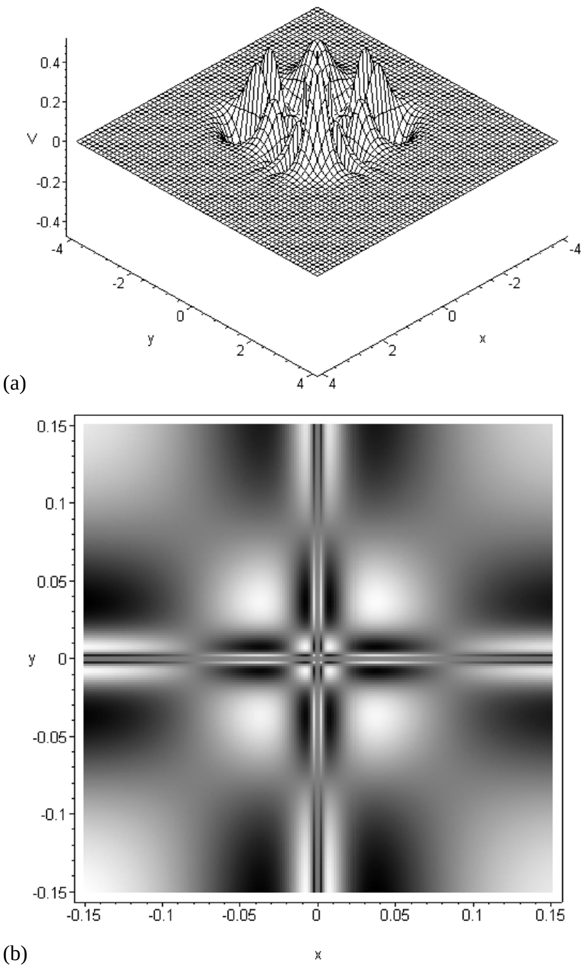
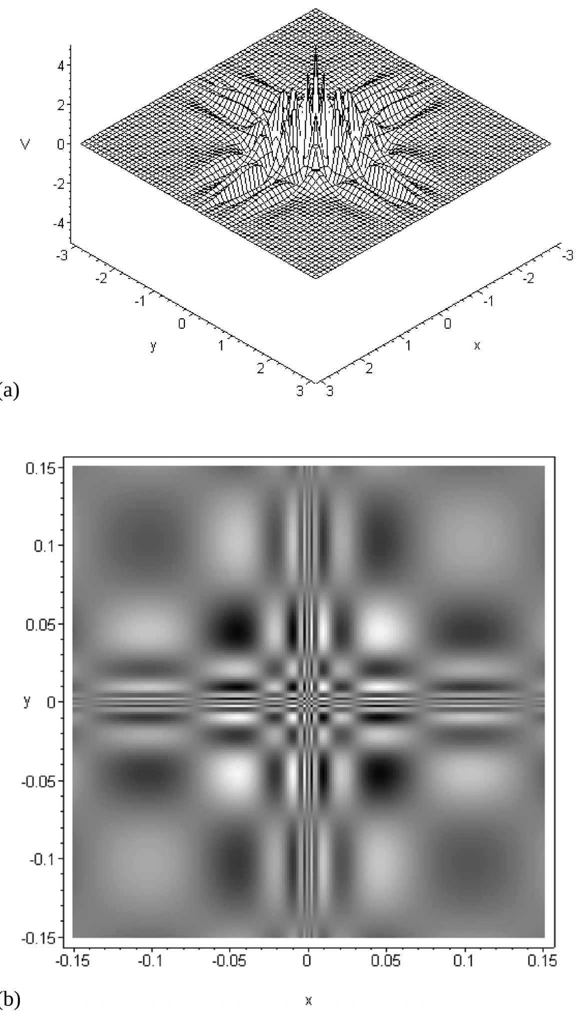


Fig. 1. Plot of the fractal dromion solution of the (2+1)-dimensional HBK system for the field quantity V shown by (13b) with the conditions (14a), (14b) and (14c). (a): The localized structure of the fractal dromion. (b): The density plot of the dromion in the range $\{x=[-0.15,0.15], y=[-0.15,0.15]\}$.The same picture (except the scales) can be found at infinitely many smaller ranges, i.e., $\{x=[-0.005,0.005], y=[-0.005,0.005]\}$, $\{x=[-0.0002,0.0002], y=[-0.0002,0.0002]\},\ldots$

Figure 1 shows the special dromion solution (13b) with the conditions (14a), (14b) and (14c) at $t=0$. The localized property of the dromion is revealed in Figure 1a. Figure 1b is a density plot of the fractal structure of the dromion solution in the range $\{x=[-0.15,0.15], y=[-0.15,0.15]\}$. It is interesting that, if we enlarge the small area at the centre of Fig. 1b, i. e., $\{x=[-0.005,0.005], y=[-0.005,0.005]\}$, $\{x=[-0.0002,0.0002], y=[-0.0002,0.0002]\},\ldots$, we find the same pictures as in Figure 1b.



(a) (b)

Fig. 2 (a): A fractal lump structure for the field quantity V with the conditions (15a), (15b), and (15c), (b): a density plot of the fractal lump related to Fig. 2 (a) at the region $\{x=[-0.15,0.15], y=[-0.15,0.15]\}$.

It is also known that in high dimensions, such as the Nizhnik-Novikov-Vesselov (NNV) equations and the ANNV (asymmetric NNV) equations, a special type of localized structure, which is called lump solution (algebraically localized in all directions), has been formed by rational functions. This localized coherent soliton structure is another type of significant localized excitation. If we select the functions α_1, α_2 and β of the field quantity V in (13b) appropriately, we can find some types of lump solutions with fractal behavior.

Figure 2a shows a fractal lump structure for the field quantity V given by (13b) at $t=0$, where α_1, α_2 and β in solution (13b) are selected as

$$\alpha_1(y)=1, \tag{15a}$$

$$\alpha_2 = 1+\frac{|y|}{1+(y)^4}\Big\{\sin\Big[\ln\big((y)^2\big)\Big] + \cos\Big[\ln\big((y)^2\big)\Big]\Big\}^2, \tag{15b}$$

$$\beta = 1+\frac{|x-\gamma t|}{1+(x-\gamma t)^4}\Big\{\sin\Big[\ln\big((x-\gamma t)^2\big)\Big] + \cos\Big[\ln\big((x-\gamma t)^2\big)\Big]\Big\}^2. \tag{15c}$$

From Fig. 2a, we can see that the solution is localized in all directions. Near the center there are infinitely many peaks which are distributed in a fractal manner. In order to investigate the fractal structure of the lump, we should look at the structure more carefully. Figure 2b presents a density plot of the structure of the fractal lump in the region $\{x=[-0.15,0.15], y=[-0.15,0.15]\}$. More detailed studies will show us the self-similar structure of the lump. For example, if we enlarge the small area at the centre of Fig. 2b, i. e., $\{x=[-0.0005,0.0005], y=[-0.0005,0.0005]\}$, $\{x=[-0.00002,0.00002], y=[-0.00002,0.00002]\},\ldots$, and so on, we can find a totally similar structure to that plotted in Fig. 2b.

3.2. Compacton Solutions and Their Interaction Behavior

It is well known that, in addition to the continuous localized excitations in (1+1)-dimensional nonlinear systems, some types of significant weak solutions, such as the compacton and peakon, have attracted much attention of both mathematicians and physicists. The so-called (1+1)-dimensional compacton solutions, which describe the typical (1+1)-dimensional soliton solutions with finite wavelength when the nonlinear dispersion effects are included, were first given by Rosenau and Hyman [16]. While the so-called peakon solution ($u=c\exp(-|x-ct|)$) refers to a weak solution of the celebrated (1+1)-dimensional Camassa-Holm equation

$$u_t+2ku_x-u_{xxt}+3uu_x=2u_xu_{xx}+uu_{xxx}, \tag{16}$$

first given by Camassa and Holm [17].

Because of the arbitrary functions in (13), we can find some types of multiple compacton solutions by selecting the arbitrary functions appropriately. For instance, if we fix the functions $\alpha_2 = 1$, β, and α_1 as

$$\begin{aligned}
&\beta = c_0, \text{ if } x+v_it \le x_{0i} - \frac{\pi}{2k_i},\\
&\beta = c_0 + \sum_{i=1}^{M} a_i \sin\left(k_i(x+v_it-x_{0i})\right) + a_i,\\
&\qquad \text{if } x_{0i} - \frac{\pi}{2k_i} < x+v_it \le x_{0i} + \frac{\pi}{2k_i},\\
&\beta = c_0 + \sum_{i=1}^{M} 2a_i, \text{ if } x+v_it > x_{0i} + \frac{\pi}{2k_i},
\end{aligned} \tag{17}$$

$$\begin{aligned}
&\alpha_1 = 0, \text{ if } y \le y_{0j} - \frac{\pi}{2l_j},\\
&\alpha_1 = \sum_{j=1}^{N} b_j \sin\left(l_j(y-y_{0j})\right) + b_j,\\
&\qquad \text{if } y_{0j} - \frac{\pi}{2l_j} < y \le y_{0j} + \frac{\pi}{2l_j},\\
&\alpha_1 = \sum_{j=1}^{N} 2b_j, \text{ if } y > y_{0j} + \frac{\pi}{2l_j}.
\end{aligned} \tag{18}$$

where c_0, a_i, k_i, v_i, b_j, l_j, x_{0i} and y_{0j} are all arbitrary, then the field quantity V with (17) and (18) becomes a multi-compacton solution.

When selecting $M = 2$, $N = 2$, $c_0 = 20$, $a_1 = -1.2$, $a_2 = -1$, $b_1 = b_2 = 1$, $k_1 = k_2 = 1$, $v_1 = -1$, $v_2 = 3$, $l_1 = l_2 = 1$, $x_{01} = x_{02} = 0$, $y_{01} = 0$, $y_{02} = 5$, then we can obtain a four-compacton structure for the (2+1)-dimensional HBK system.

Figures 3A(a – f) show the evolution behavior of interaction between four compactons. We see that the interaction among four compactons exhibits a new phenomenon, that is, the interaction among four compactons is non-elastic, but four compactons do not completely exchange their shapes after interaction. Figures 3B(a – f) show the interacting evolution behavior of less symmetric cases of the compactons. We find that the symmetry and degrees of inelasticity of the compacton solutions differ and are determined by the parameters k, l, moreover, when $k_1 \ne k_2$, we see in Fig. 3B that the compacton cannot complete the superposition at $t = 0$. Similar properties also occur in peakon solutions of the HBK system.

3.3. Peakon Solutions and Their Interaction Behavior

Similarly, considering the arbitrariness of the functions α_1, α_2 and β in (13), we can construct the peakon solution of the (2+1)-dimensional HBK system by selecting appropriate functions. For instance, when $\alpha_2 = -1$, β and α_1 are taken as the following simple form:

$$\begin{aligned}
&\beta = c_0 + \sum_{i=1}^{M} d_i \exp(m_ix - v_it + x_{0i}),\\
&\quad \text{if } m_ix - v_it + x_{0i} \le 0,\\
&\beta = c_0 + \sum_{i=1}^{M} -d_i \exp(-m_ix + v_it - x_{0i}) + 2d_i,\\
&\quad \text{if } m_ix - v_it + x_{0i} > 0,
\end{aligned} \tag{19}$$

$$\begin{aligned}
&\alpha_1 = \sum_{j=1}^{N} e_j \exp(n_jy + y_{0j}), \text{ if } n_jy + y_{0j} \le 0,\\
&\alpha_1 = \sum_{j=1}^{N} -e_j \exp(-n_jy - y_{0j}) + 2e_j,\\
&\quad \text{if } n_jy + y_{0j} > 0,
\end{aligned} \tag{20}$$

where c_0, d_i, m_i, v_i, e_j, n_j, x_{0i} and y_{0j} are all arbitrary constants, the field quantity V with (19) and (20) becomes a multi-peakon solution. If we select $M = 2$, $N = 2$, $c_0 = 5000$, $d_1 = d_2 = 1$, $m_1 = m_2 = 1$, $v_1 = -1$, $v_2 = 2$, $e_1 = e_2 = 1$, $n_1 = n_2 = 1$, $x_{01} = 4$, $x_{02} = -4$, $y_{01} = 4$, $y_{02} = -4$, we obtain a four-peakon structure for the (2+1)-dimensional HBK system.

Figures 4(a – f) show the evolution behavior of interaction among four peakons. We find that the interaction among the four peakons exhibits a new phenomenon, that is, the interaction among the four peakons is not completely elastic, but the four peakons may completely exchange their shapes after interaction.

3.4. Folded Solitary Waves, Foldons and Their Interaction Behavior

Because the real natural phenomena are very complicated, in various cases it is even impossible to describe the natural phenomena by single-valued functions. For instance, in the real natural world, there exist very complicated folded phenomena such as the folded protein [18], folded brain and skin surface, and many other kinds of folded biologic systems [19]. The simplest multi-valued (folded) waves may be the bubbles on (or under) a fluid surface. Various ocean waves are folded waves also.

Now we discuss a new type of folded localized excitation for the (2+1)-dimensional HBK system. As is known, the simplest foldons are the so-called loop solitons [7], which can be found in many (1+1)-dimensional integrable systems [7] and have been applied in some possible physical fields like the string

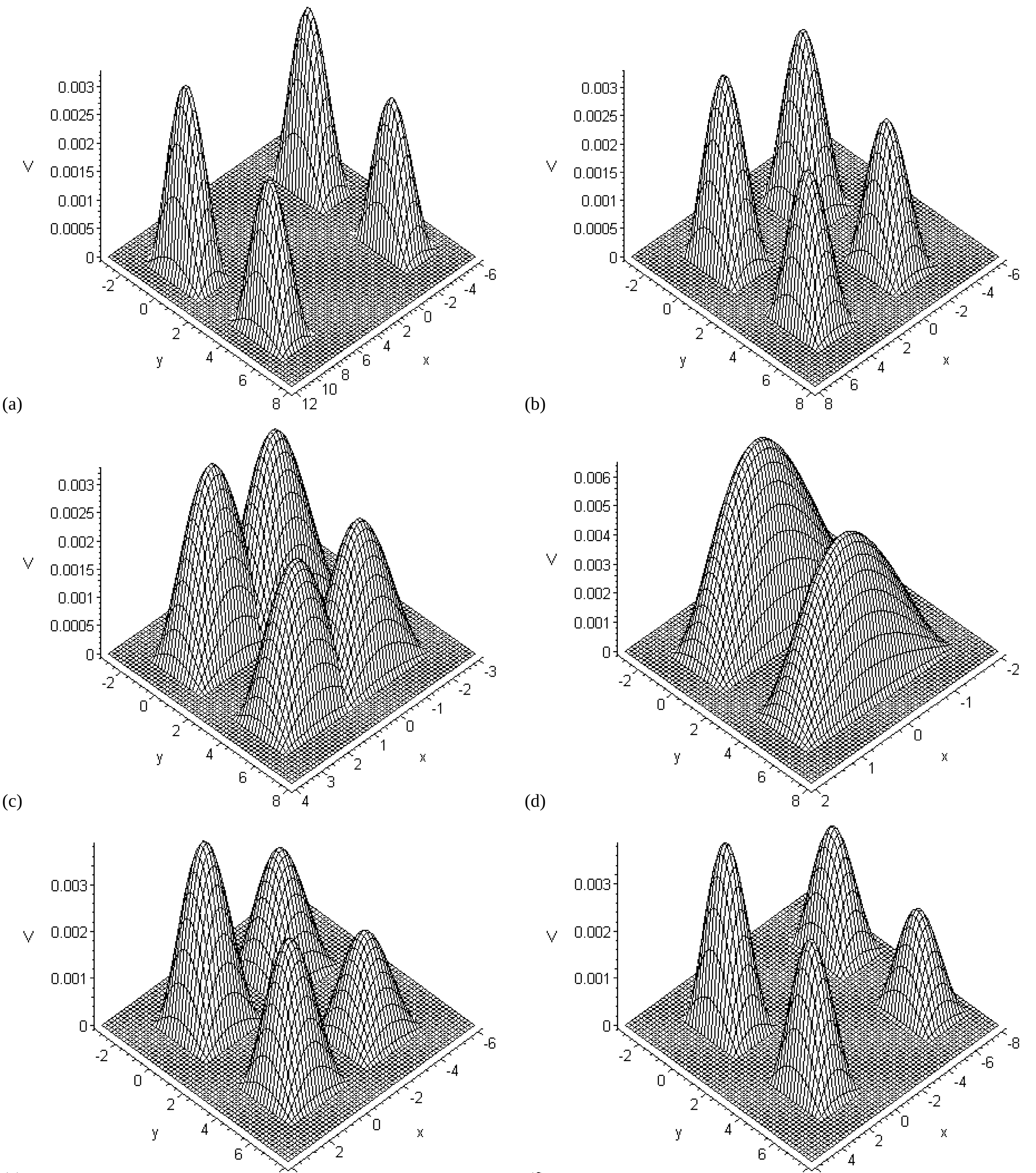


Fig. 3A. Evolution plot of a four-compacton solution determined by (13b) at (a): $t = -3$, (b): $t = -1.5$, (c): $t = -0.6$, (d): $t = 0$, (e): $t = 1$, and (f): $t = 2$ with (17) and (18).

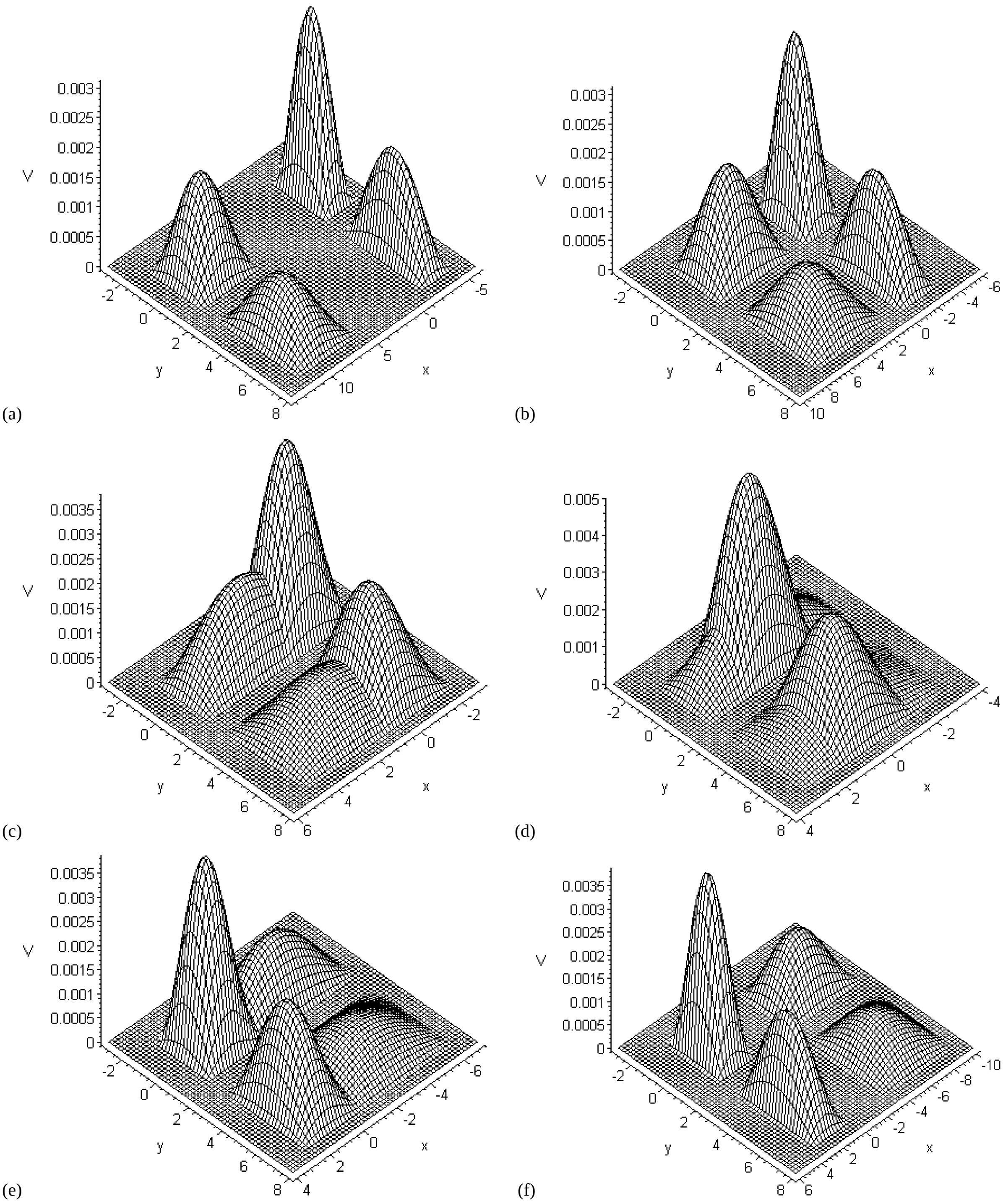


Fig. 3B. Evolution plot of a less symmetric case of the four-compacton solution determined by (13b) under the same condition as Fig. 3A, but with $k_1 = 1$, $k_2 = 1/2$, $l_1 = 1$, and $l_2 = 2/3$.

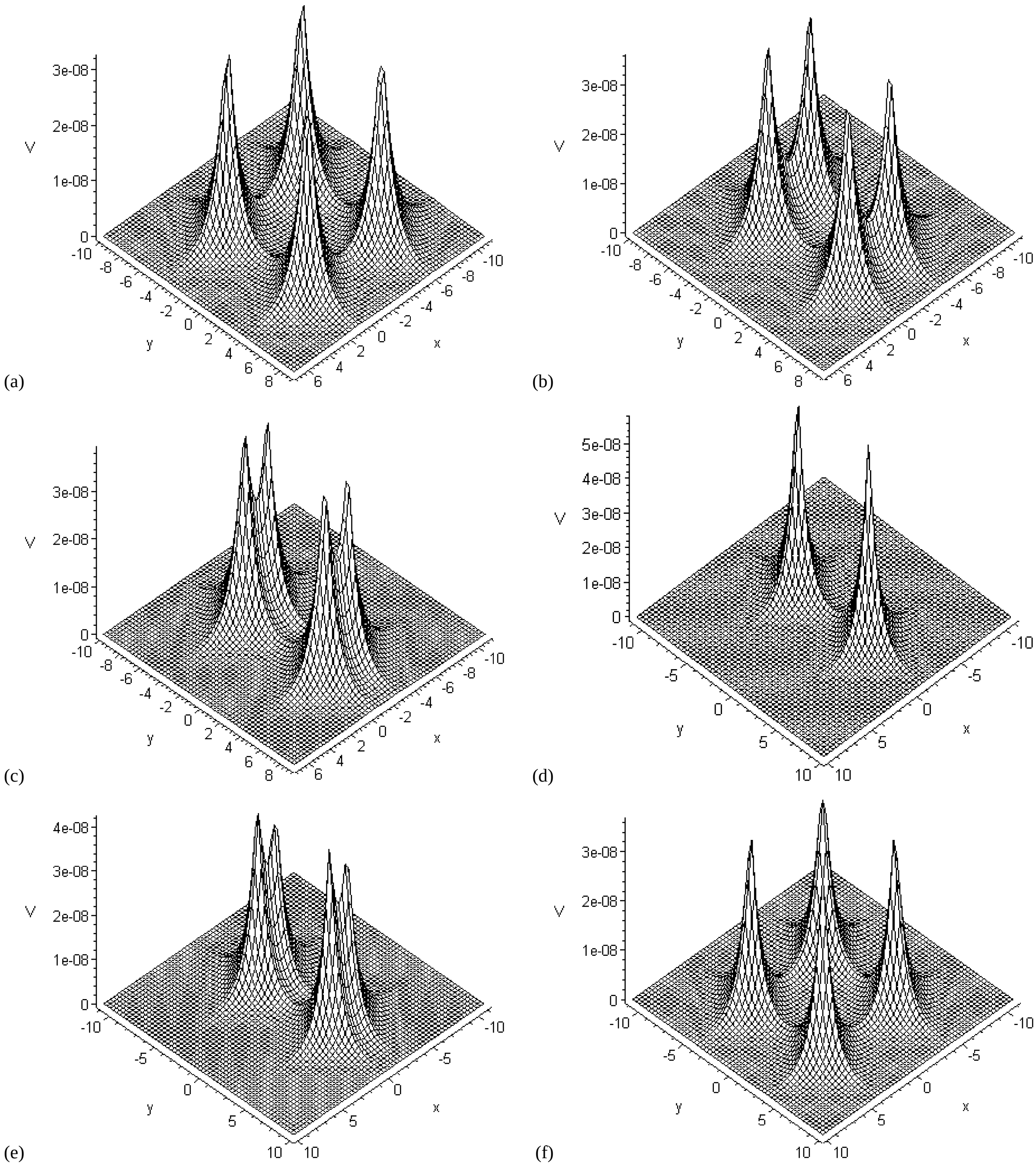


Fig. 4. Evolution plot of a four-peakon solution determined by (13b) at (a): $t = -5$, (b): $t = -4$, (c): $t = -3.4$, (d): $t = -2.65$, (e): $t = -2$, and (f): $t = 0$ with (19) and (20).

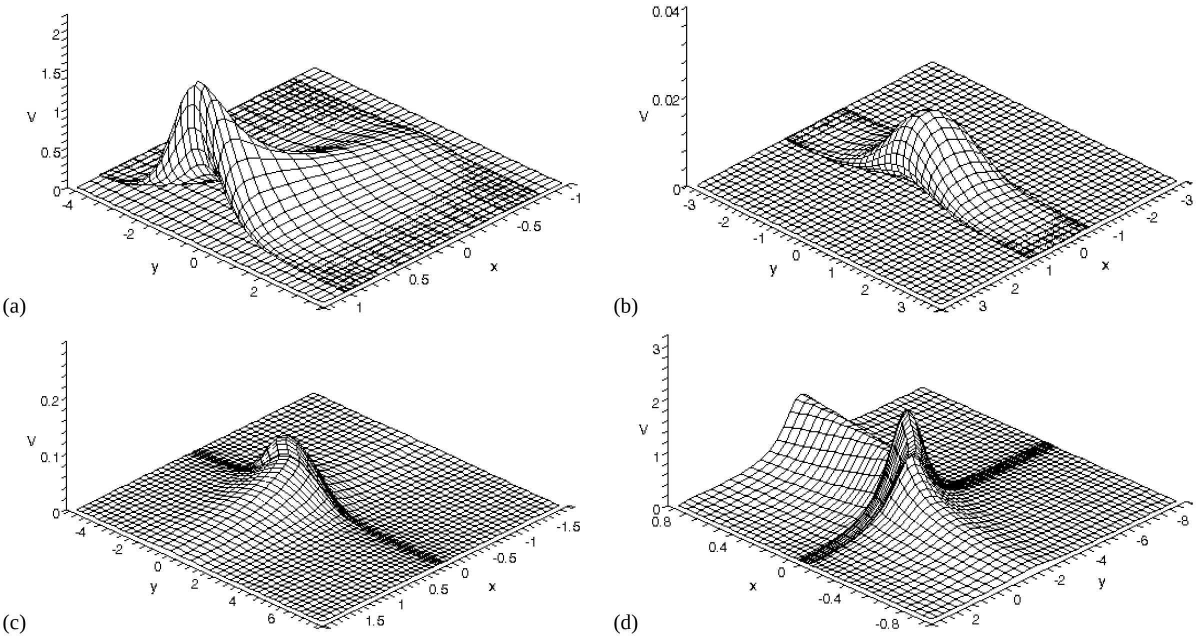


Fig. 5 Four typical folded solitary waves for the field quantity V, determined by (13b) at $t = 0$ with (23) – (26) for (a) the "tent" shape, (b) the "worm" shape, (c) the "worm-dromion" shape, and (d) the "worm-solitoff" shape folded solitary wave, respectively.

interaction with external field [20], quantum field theory [21], and particle physics [22]. However, how to find some folded localized excitations and/or foldons in higher-dimensional physical models is still open.

In order to construct interesting folded localized excitations and/or foldons for the field quantity V, we should introduce some suitable multi-valued functions. For example

$$\beta_x = \sum_{j=1}^{M} U_j(\xi + w_j t), \quad x = \xi + \sum_{j=1}^{M} X_j(\xi + w_j t), \tag{21}$$

where U_j and X_j are localized excitations with the properties $U_j(\pm\infty) = 0, X_j(\pm\infty) = \text{const}$. From (21) one can knows that ξ may be a multi-valued function in some suitable regions of x by selecting the functions X_j appropriately. Therefore, the function β_x, which is obviously an interaction solution of M localized excitations because of the property $\xi\,|_{x\to\infty} \to \infty$, may be a multi-valued function of x in these areas, though it is a single-valued functions of ξ. Actually, most of the known multi-loop solutions are a special situation of (21). Similarly, we also treat the function $\alpha_1(y)$ in this way:

$$\alpha_{1_y} = \sum_{j=1}^{N} V_j \eta, \quad y = \eta + \sum_{j=1}^{N} Y_j \eta. \tag{22}$$

In Fig. 5, four typical folded solitary waves are plotted for the field quantity V determined by (13b) with the function selections

$$\begin{aligned}
&\beta_x = -\text{sech}^2(\xi + wt),\\
&\beta = \frac{2\sinh(\xi + wt)}{3\cosh(\xi + wt)} + \frac{5\sinh(\xi + wt)}{6\cosh^3(\xi + wt)} + 0.9,\\
&x = \xi - 2.5\tanh(\xi + wt),\\
&\alpha_{1_y} = -\text{sech}^2(\eta), \quad \alpha_1 = -\frac{\sinh(\eta)}{\cosh(\eta)}, \quad y = \eta.
\end{aligned} \tag{23}$$

$$\begin{aligned}
&\beta_x = -\text{sech}^2(\xi + wt),\\
&\beta = \frac{2\sinh(\xi + wt)}{3\cosh(\xi + wt)} + \frac{5\sinh(\xi + wt)}{6\cosh^3(\xi + wt)} + 7,\\
&x = \xi - 2.5\tanh(\xi + wt),\\
&\alpha_{1_y} = -\text{sech}^2(\eta), \quad \alpha_1 = -\frac{\sinh(\eta)}{\cosh(\eta)}, \quad y = \eta.
\end{aligned} \tag{24}$$

$$\beta_x = -10\text{sech}^2(\xi + wt),$$

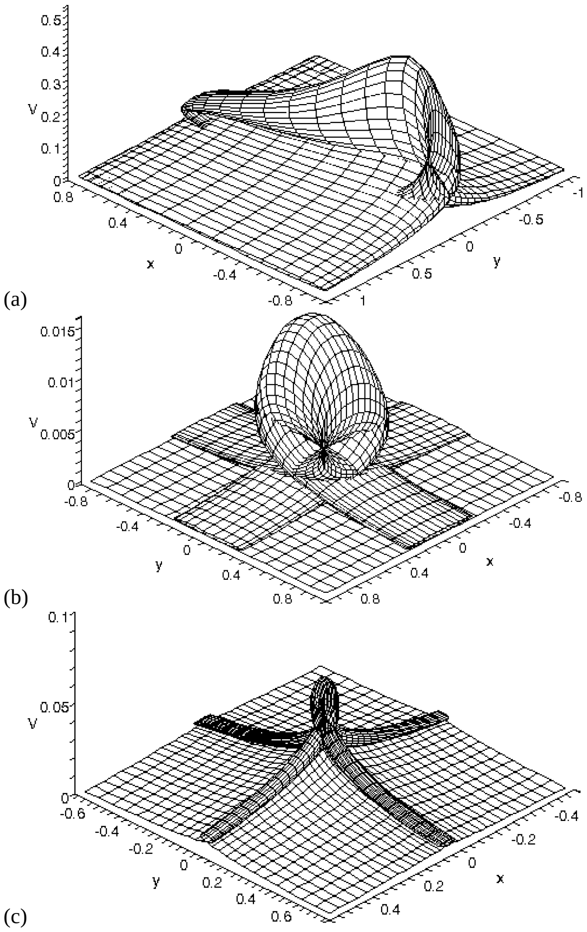


Fig. 6. Three typical folded solitary waves for the field quantity V determined by (13b) at $t = 0$ with (27)–(29) shown in (a), (b), and (c), respectively.

$$\beta = -\frac{7\sinh(\xi + wt)}{3\cosh(\xi + wt)} + \frac{23\sinh(\xi + wt)}{6\cosh^3(\xi + wt)} + 9,$$
$$x = \xi - 1.15\tanh(\xi + wt),$$
$$\alpha_{1_y} = -\operatorname{sech}^2(\eta), \quad \alpha_1 = -\frac{5\sinh(\eta)}{3\cosh(\eta)} - \frac{\sinh(\eta)}{3\cosh^3(\eta)},$$
$$y = \eta + \tanh(\eta). \tag{25}$$
$$\beta_x = -\operatorname{sech}^2(\xi + wt),$$
$$\beta = -\frac{7\sinh(\xi + wt)}{30\cosh(\xi + wt)} + \frac{23\sinh(\xi + wt)}{60\cosh^3(\xi + wt)} + 0.95,$$
$$x = \xi - 1.15\tanh(\xi + wt),$$
$$\alpha_{1_y} = -\operatorname{sech}^2(\eta), \quad \alpha_1 = -\frac{5\sinh(\eta)}{3\cosh(\eta)} - \frac{\sinh(\eta)}{3\cosh^3(\eta)},$$
$$y = \eta + \tanh(\eta). \tag{26}$$

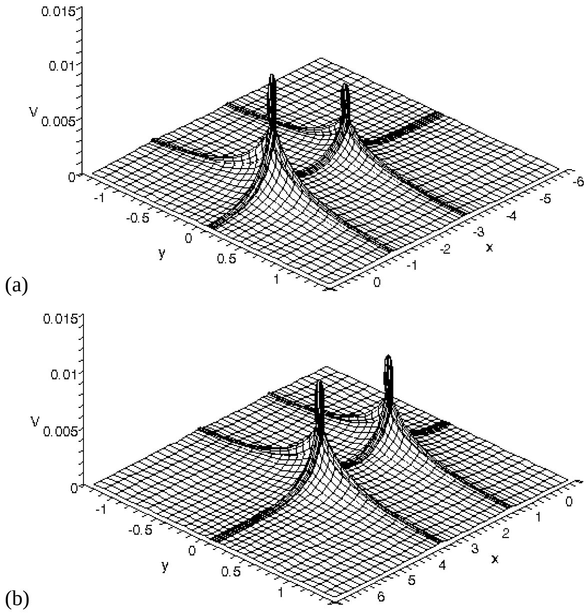


Fig. 7. Pre- and post-interaction of two folded solitary waves at time (a) $t = -4.5$, and (b) $t = 4.5$ for the field quantity V determined by (13b) with the selections (30).

Figure 6 shows other three typical folded solitary waves for the field quantity V determined by (13b) with the function selections (27)–(29). However, the parameters are chosen such that both β and α_1 are multi-valued.

$$\beta_x = -\operatorname{sech}^2(\xi + wt),$$
$$\beta = -\frac{\sinh(\xi + wt)}{15\cosh(\xi + wt)} + \frac{7\sinh(\xi + wt)}{15\cosh^3(\xi + wt)} + 0.9,$$
$$x = \xi - 1.4\tanh(\xi + wt),$$
$$\alpha_{1_y} = -\operatorname{sech}^2(\eta), \quad \alpha_1 = \frac{2\sinh(\eta)}{3\cosh(\eta)} + \frac{5\sinh(\eta)}{6\cosh^3(\eta)},$$
$$y = \eta - 2.5\tanh(\eta). \tag{27}$$
$$\beta_x = -\operatorname{sech}^2(\xi + wt),$$
$$\beta = \frac{\sinh(\xi + wt)}{15\cosh(\xi + wt)} + \frac{8\sinh(\xi + wt)}{15\cosh^3(\xi + wt)} + 7,$$
$$x = \xi - 1.6\tanh(\xi + wt),$$
$$\alpha_{1_y} = -\operatorname{sech}^2(\eta), \ \alpha_1 = \frac{\sinh(\eta)}{15\cosh(\eta)} + \frac{8\sinh(\eta)}{15\cosh^3(\eta)},$$
$$y = \eta - 1.6\tanh(\eta). \tag{28}$$
$$\beta_x = -\operatorname{sech}^2(\xi + wt),$$

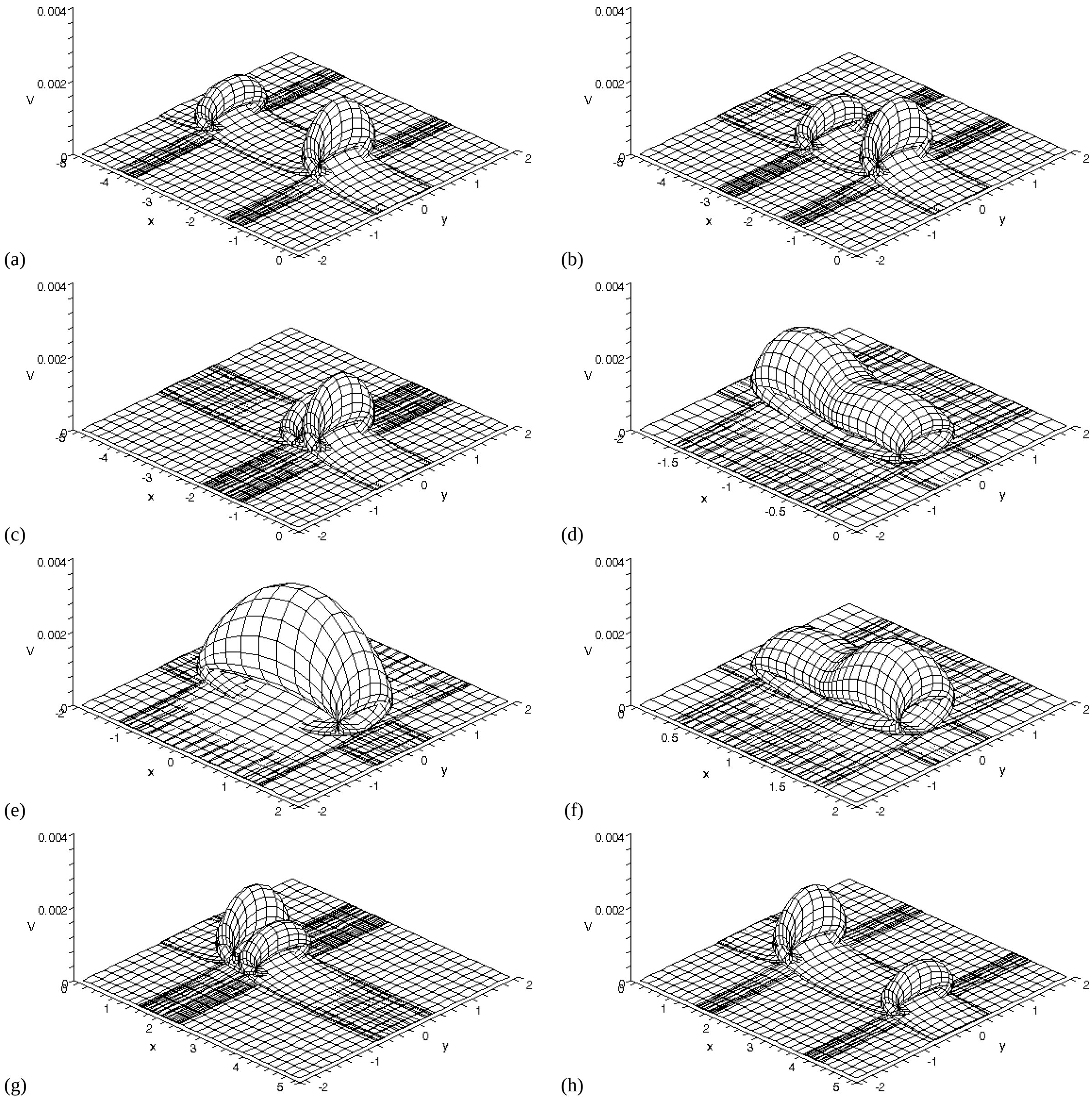


Fig. 8. Evolution plots of two foldons for the field quantity V determined by (13b) with the selections (31) at time (a) $t=-5.5$, (b) $t=-4.5$, (c) $t=-3.5$, (d) $t=-2$, (e) $t=0$, (f) $t=2$, (g) $t=3.5$, and (h) $t=5.5$.

$$\beta = -\frac{7\sinh(\xi+wt)}{30\cosh(\xi+wt)} + \frac{23\sinh(\xi+wt)}{60\cosh^3(\xi+wt)} + 3,$$

$$x = \xi - 1.15\tanh(\xi+wt),$$

$$\alpha_{1_y} = -\mathrm{sech}^2(\eta),\ \alpha_1 = -\frac{7\sinh(\eta)}{30\cosh(\eta)} + \frac{23\sinh(\eta)}{60\cosh^3(\eta)},$$

$$y = \eta - 1.15\tanh(\eta). \tag{29}$$

Figure 7 is a pre- and post-interaction plot of two folded solitary waves for the field quantity V determined by (13b) with the selections

$$\beta_x = -12\mathrm{sech}^2(\xi) - 10\mathrm{sech}^2(\xi - t),$$

$$x = \xi - 1.15\tanh(\xi) - 1.15\tanh(\xi - t),$$

$$\alpha_{1_y} = -\mathrm{sech}^2(\eta), \quad y = \eta - 1.15\tanh(\eta). \tag{30}$$

Figure 8 shows evolution plots of two foldons for the field quantity V determined by (13b) with the selections

$$\beta_x = -\frac{4}{5}\mathrm{sech}^2(\xi) - \frac{1}{2}\mathrm{sech}^2(\xi - t),$$
$$x = \xi - 1.5\tanh(\xi) - 1.5\tanh(\xi - t),$$
$$\alpha_{1_y} = -\mathrm{sech}^2(\eta), \quad y = \eta - 2\tanh(\eta). \tag{31}$$

4. Summary

In summary, starting from the quite general solution for the (2+1)-dimensional HBK system, four kinds of new localized excitations (fractal-dromion and lump, peakon, compacton and foldon (and/or folded solitary wave)) can be constructed by selecting arbitrary functions appropriately, like dromions, lumps, ring solitons, breathers, instantons, solitoffs, and chaotic patterns. The interactions among peakons, compactons and foldons exhibit interesting novel features not found in one-dimensional solitons. Since the excitation (13b) is a "universal" formula for some (2+1)-dimensional physical models, and the (2+1)-dimensional HBK system presents several significant physical models, all the discussions in this work are valid for the above-mentioned physical systems. This work is only a beginning attempt. Further study to find localized excitations like new types of folded solitary waves, foldons and their application is necessary.

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